# Title

Author
Institution

## A Block)

Define polynomials $F_n(x)$ by the following:

$$F_n(x) = (-1)^n(1+x^2)^2 e^{\frac{x^2}{2}} \frac{d}{dx}\left((1+x^2)^{-1}\frac{d^{n-1}}{dx^{n-1}}e^{-\frac{x^2}{2}}\right), \quad n = 1,2,3,\ldots,$$

The polynomials $\{F_n(x)\}$ arose when studying an example of potential of the anharmonic oscillator which has an equidistant spectrum.

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + q(x), \text{ where } q(x) = \frac{1}{8}x^2 + \frac{2}{1+x^2} - \frac{4}{(1+x^2)^2} + \frac{1}{3}$$

and satisfy the following:

$$F_n(x) = He_{n+2}(x) + 2(n+2)He_n(x) + (n+2)(n-1)He_{n-2}(x)$$

where $He_n(x)$ is the $n$-th monic Hermite polynomial.

Some math

$$1 + 1 = 2$$

## Another Block

The polynomials $\{F_n(x)\}$ are orthogonal in the sense that

$$\int_{-\infty}^{\infty} F_n(x)F_m(x)\frac{e^{-\frac{x^2}{2}}}{(1+x^2)^2}dx = (n-1)(n-1)!(2\pi)^{\frac{1}{2}}\delta_{n,m}$$

for any nonnegative integers $n$ and $m$, where $\delta_{n,m}$ is the Kronecker delta. The $\{F_n(x)\}$ are solutions to the following differential equation:

$$(1+x^2)\left(\frac{d^2F_n(x)}{dx^2} - x\frac{dF_n(x)}{dx} + (n+2)F_n(x)\right) = 4x\frac{dF_n(x)}{dx}$$

## Another Block

**Exceptional Orthogonal Polynomials** are families of polynomials which

1. Form a sequence of orthogonal polynomials with finitely many missing degrees.
2. Are eigenfunctions of a second order differential operator.
3. Are complete in their corresponding weighted $L^2$ space.

## Example

Here's an example.

## A Block

Let $\{P_n(x)\}$ be a monic OPS with respect to a symmetric measure $d\mu(x)$ supported on $\mathbb{R}$. We will consider a sequence of polynomials $R_n(x)$ defined as

$$R_n(x) := P_{n+2}(x) + A_nP_n(x) + B_nP_{n-2}(x), \quad R_0(x) := 1,$$

for sequences $\{A_n\}$ and $\{B_n\}$ of real numbers such that

1. $R_n'(i) = 0$ for all $n \geq 1$
2. $\int_{-\infty}^{\infty} R_0(x)R_n(x)\frac{d\mu(x)}{(1+x^2)^2} = 0$ for all $\quad n \geq 1$
3. $\int_{-\infty}^{\infty} R_1(x)R_n(x)\frac{d\mu(x)}{(1+x^2)^2} = 0$ for all $n \geq 2$

Enumerate some math:

1. $R_n(x)$ polynomial of degree $n+2$.
2. $R_n'(-i) = 0$.
3. The sequence $\{R_n(x)\}$ does not contain a degree 1 or 2 polynomial.

## Alert Block

The polynomials $R_n(x)$ are orthogonal with respect to $\frac{d\mu(x)}{(1+x^2)^2}$ i.e.

$$\int_{-\infty}^{\infty} R_n(x)R_m(x)\frac{d\mu(x)}{(1+x^2)^2} = 0$$

for $n \neq m$ and is nonzero for $m = n$.

Let $\{P_n(x)\}$ be a family of symmetric orthogonal polynomials. Then they satisfy a 3-term recurrence relation which corresponds to the monic Jacobi matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ a_1 & 0 & 1 & \\ 0 & a_2 & 0 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

i.e

$$J\mathbf{P}(x) = x\mathbf{P}(x),$$

where $\mathbf{P}(x) = (P_0(x), P_1(x), P_2(x), \ldots)^\top$.

## Another Alert Block

Big Theorem

More text.

## Example

Let $\hat{T}_n(x)$ denote the $n$-th monic Chebyshev polynomial.

## Alert Block in an Example Block

There exists sequences of real numbers $\{A_n\}$ and $\{B_n\}$ such that the polynomials $\mathcal{R}_n(x) := \hat{T}_{n+2}(x) + A_n\hat{T}_n(x) + B_n\hat{T}_{n-2}(x)$ satisfy

1. $\mathcal{R}_n'(i) = 0$ for all $n = 1, 2, \ldots$
2. $\int_{-1}^{1} \frac{\mathcal{R}_0(x)\mathcal{R}_n(x)}{\sqrt{1-x^2}(1+x^2)^2}dx = 0$ for all $n = 1, 2, \ldots$
3. $\int_{-1}^{1} \frac{\mathcal{R}_1(x)\mathcal{R}_n(x)}{\sqrt{1-x^2}(1+x^2)^2}dx = 0$ for all $n = 2, 3, \ldots$

where $\mathcal{R}_0(x) := 1$.

Below you can put a picture:

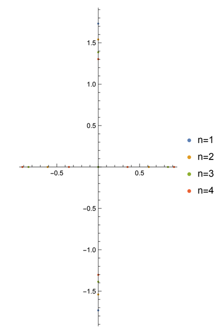

Figure: Caption 1

## References


- V.E. Adler, *On a modification of Crum's method*. Teoret. Mat. Fiz. 101 (1994), no. 3, 323–330 (Russian); translation in Theoret. and Math. Phys. 101 (1994), no. 3, 1381–1386 (1995)
- R. Bailey and M. Derevyagin. *DEK-type orthogonal polynomials and a modification of the Christoffel formula*. J. Comput. Appl. Math. 438 (2024), Paper No. 115561.
- S.Yu. Dubov, V.M. Eleonskii, and N.E Kulagin, *Equidistant Spectra of Anharmonic Oscillators*, Soviet Phys. JETP 75 (1992), no. 3, 446–451; translated from Zh. Èksper. Teoret. Fiz. 102 (1992), no. 3, 814–825 (Russian).